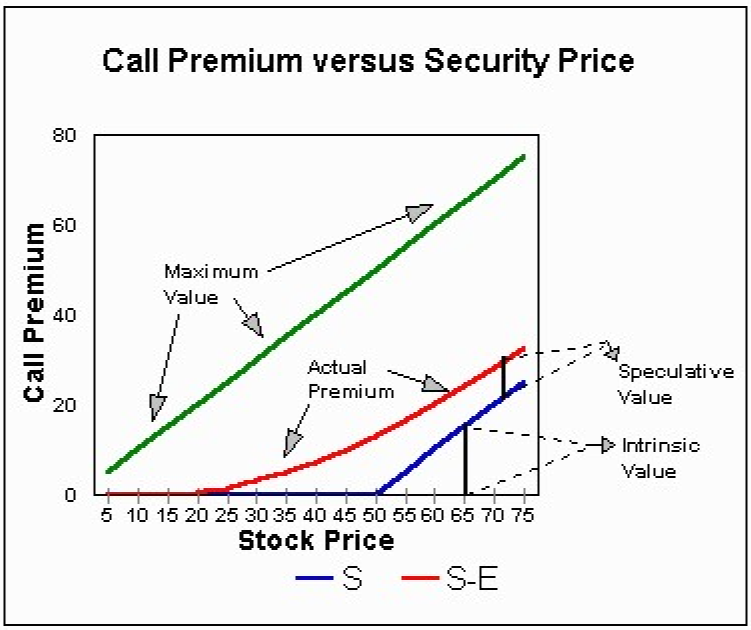
LECTURE 6

Chapter 5 – Black Sholes Option Pricing Model

BLACK-SHOLES OPTION PRICING MODEL

<u>History</u>:
Brownian Motion (1820s) → Louis Bachelier (1900) → Albert Einstein (Theory of Relativity (1910's) → Norbert Wiener (Stochastic Calculus) → Black Sholes Merton Model



The **Black–Scholes** model is a mathematical description of financial markets and derivative investment instruments. The model develops partial differential equations whose solution, the **Black–Scholes formula**, is widely used in the pricing of European-style options.

$$\mathbf{C = \underbrace{S\, e^{-\delta t}\, N(d1)}_{\text{Stock Price}} - \underbrace{X\, e^{-it}\, N(d2)}_{\text{Strike Price at PV (X) compounding at } e}}$$

$$\underbrace{d1 = [\ln (S/X) + (i - \delta + \sigma^2 / 2)\, t\,] / \sigma \sqrt{t}}$$

Measures the stock volatility until expiration

$$\underbrace{d2 = d1 - \sigma \sqrt{t}}$$

Measures the Option price volatility until expiration

A little Mathematical Review first:

- **Logarithms and e**: The two most commonly used base numbers are 10 and e, where e is a special number approximately equal to 2.71828. i.e. $Log_{10}25 = 1.3979$ (the log of base 10 of 25) or $10^{1.3979} = 25$
- **Log_e = Ln: - explaining e compounding:** Consider the investment of $100 at 6% will yield 106, at 6% semiannually will yield 106.09, quarterly will yield 106.13…. if continue compounding – close to infinity times the maximum limit will reach 2.71828 times the investment or $e^{0.06} = 1.06183655$
- **Normal Distribution or Normal Probability Distribution:** The bell shaped graph – approximately 68% of the observations in a sample drawn from normal distribution will occur within one standard deviation (σ) of the expected value.
- **Z Statistic:** This is the standard normal random variable (-3 to +3) determines the probability (positive value from 0 to 1).

Black-Scholes Model - Definition
A mathematical formula designed to price an option as a function of certain variables- generally stock price, striking price, volatility, time to expiration, dividends to be paid, and the current risk-free interest rate.

Black-Scholes Model - Introduction
The Black-Scholes model is a tool for equity options pricing. Prior to the development of the Black-Scholes Model, there was no standard options pricing method and nobody can put a fair price to charge for options. The Black-Scholes Model turned that guessing game into a mathematical science which helped develop the options market into the lucrative industry it is today. Options traders compare the prevailing option price in the exchange against the theoretical value derived by the Black-Scholes Model in order to determine if a particular option contract is over or under valued, hence assisting them in their options trading decision. The Black-Scholes Model was originally created for the pricing and hedging of European Call and Put options as the American Options market, the CBOE, started only 1 month before the creation of the Black-Scholes Model. The difference in the pricing of European options and American options is that options pricing of European options do not take into consideration the possibility of early

exercising. American options therefore command a higher price than European options due to the flexibility to exercise the option at anytime. The classic Black-Scholes Model does not take this extra value into consideration in its calculations.

The value of a call option in terms of the Black–Scholes parameters is:

$$C = Se^{-\delta t} N(d1) - Xe^{-it} N(d2)$$

$$d1 = [\ \ln(S/X) + (i + - \delta + \sigma^2 / 2)\ t / [\ \sigma \sqrt{t}]$$

$$d2 = d1 - \sigma \sqrt{t}$$

- N(•) is the cumulative distribution function of the standard normal distribution
- T - t is the time to maturity
- S is the spot price of the underlying asset
- X is the strike price
- r is the risk free rate (annual rate, expressed in terms of continuous compounding)
- σ is the volatility in the log-returns of the underlying

Example

DCRB June 125 Call – T-Bill at 4.46%

S=125.94, X=125, i=0.0446, σ=0.83, t=0.0959

Using the Black Sholes we calculate d1=0.1743, d2=-0.0827, N(d1)=0.5692, N(d2)=0.4670 and C=13.56

Explaining the Formula:

1. Measuring the future Payoff (S-X)

   Of course, predicting the difference of S – X at expiration day should help determine the price that you are willing to pay today. Since you are pricing the premium you are willing to pay **today** you first need to present value the future price of X by the implied growth or free rate or PV(X). Since the Interest rate needs to be continuously compounded by using the **e** constant so the PV(X) or 1 /

$(1+i)^t$ can be rewritten as Xe^{-it} or the pyoff at $S - Xe^{-it}$. If there is a chance that the stock (S) will be paying a dividend (δ) by the exercise day which would affect the value of S, an adjustment needs to be made today such as the value of S is adjusted for a potential future dividend (δ) so $Se^{-\delta t}$ The payoff then is measured at

$$Se^{-\delta t} - Xe^{-it} .$$

2. <u>Adjusting for Probability and Delta Hedging N (d1) and N (d2)</u>

N(d1) and N(d2) are the probabilities of the option expiring in-the-money under the equivalent exponential martingale probability measure (numéraire = stock) and the equivalent martingale probability measure (numéraire = risk free asset), respectively. The equivalent martingale probability measure is also called the risk-neutral probability measure. Note that both of these are probabilities in a measure theoretic sense, and neither of these is the true probability of expiring in-the-money under the real probability measure. In order to calculate the probability under the real ("physical") probability measure, additional information is required.

d—the drift term in the physical measure, or equivalently, the market price of risk.

The highest stock price should lead to a higher call price. Using our previous examples suppose the stock price is \$130 instead of \$125.94. Calculating N(d1) and N(d2) it gives us 0.6171 and 0.5162, respectively, which gives a value of C of \$15.96 (running the full Black Sholes), which is higher than the previously obtained of \$13.55.

The relationship between the stock price and call price is referred to as **delta** – the probability that the delta must range between zero and one.

The delta hedge (similar to the BOPM calculating h) position is said to be neutral. For example, the stock price is \$125.94. Recall that the delta is 0.5692, so we construct a delta hedge by buying 569 shares and selling 1,000 calls. If the stock price falls by a small amount (for instance \$0.01), we shall lose \$0.01 (569) = \$5.69 on the stock. The option price, however will fall be approximately \$0.01 (0.569), or \$0.00569. Because we have 1,000 calls, the options collectively will fall by \$0.00569 (1.000) = \$5.69. Because we short the options, we gain \$5.69, which offsets the loss on the stock.

- Concept of **Gamma (γ)** – the small but important changes of Delta expressed in a formula:

$$\boxed{\textbf{Gamma } (\gamma) = e^{-(d1^{\wedge}2)/2} / S\sigma\sqrt{2\pi t}}$$

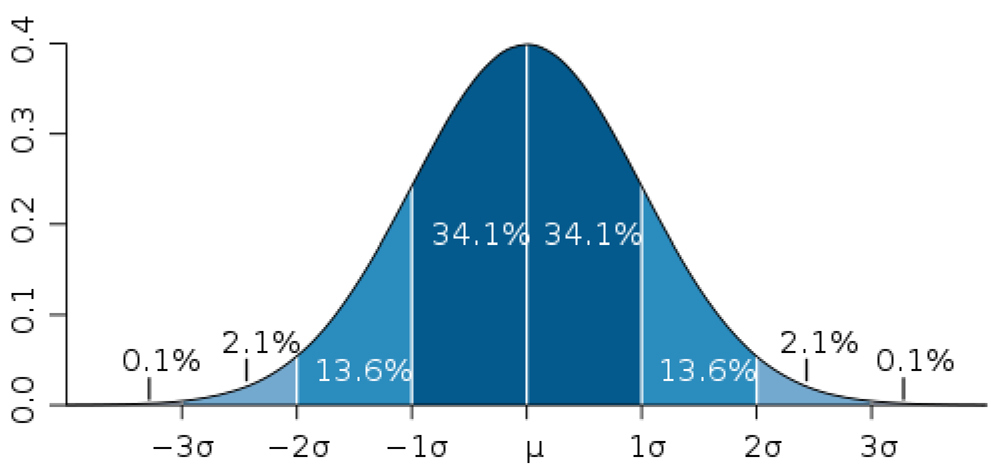
3. Risk Free Rate (i)

The risk free rate must be expressed as a continuously compounded rate. The interest rate movement will have a very small impact in the overall value of the premium. If we raise the interest rates by a very large rate - suppose rates go up from 4% to 12% or 300% - the impact of the premium will go up less than 3%. The sensitivity of the call price to the risk-free rate is called **Rho (ρ)** and is given by its formula:

$$\boxed{\textbf{Rho } (\rho) = tXe^{-it} N(d2)}$$

4. Volatility or Standard Deviation (σ):

The most significant assumption that has the most sensitivity to the price of the premium is that volatility - a measure of how much a stock can be expected to move in the near-term, is a constant over time. The measurement of the stock is calculated by the historical standard deviation. Using a perfect normal distribution graph to price the future stock price we could assign values given the movement of the stock (within 68.2% for one standard deviation or 95.4% for 2 standard deviations, and so on)



The sensitivity of the call price is very small change in volatility is called lamda (note: the text book calls it Vega):

$$\boxed{\textbf{Lamda} = St\sqrt{\ }\, e^{-(d1^{\wedge}2)/2} \,/\, \sqrt{2\pi}}$$

In our original example, the Lamda is calculated at \$15.32. This suggest that if the volatility changes by a very small amount (for example, 0.01), the call price will change by 15.32 (0.01) = 0.15. The actual call price will change if volatility went to 0.84, an increase of 0.01, would be 0.16. If the volatility increased 0.12 to 0.95, the lamda predicts that the call price would increase by 15.32 (0.12) = 1.84, when the actual call price would go to \$15.39, an increase of \$15.39-\$13.55 = \$1.84

5. Time to Expiration and Time Decay

The decrease in the value of the call as time elapses in the time value decay. The rate of time value decay is measured by the option's theta, which is given as

$$\boxed{\textbf{Theta} = [\, -S\sigma\, e^{-(d1^{\wedge}2)/2} \,/2\sqrt{2\pi t}] - iXe^{-it}\, N(d2)}$$

For the June 125, d1 is 0.1743, N(d2) is 0.4670, and theta is -68.91. Thus, in one week, the time to expiration goes from 0.0959 to 28/365 = 28/365 = 0.0767 and the option price would be predicted to change by (0.0959 – 0.0767) (-68.91) = -\$1.32. The actual price changes to \$12.16, so the change is \$12.16 - \$13.55 = -\$1.39. Of course, the theta is most accurate only for a very small change in time, but the predicted price change is still very close to the actual price change. Because time itself I not a source of risk, however, it makes no sense to worry about it, but of course, that does not mean we do not need to know the rate of time value decay.

The price of a put option is:

$$\mathbf{P = X\, e^{-it}\, N\,(1\text{-}d2) - S\, e^{-\delta t}\, N\,(1\text{-}d1)}$$

Example of a Call Option
Suppose you want to value a call option under the following circumstances

Stock Price	$S_0 = 100$
Exercise Price	X=95
Interest Rate	r= .10
Dividend Yield	$\delta = 0$
Time to expiration	T = .25 (one-quarter year)

Standard Deviation $\sigma = .50$

First calculate

$d1 = [\ln(100/95) + (.10-0 + .5^2/2).25] / [.5 \sqrt{25}] = .43$

$d2 = .43 - .5 \sqrt{.25} = .18$

Next find N (d1) and d N(d2). The normal distribution function is tabulated and may be found in many statistics books – see below. The normal distribution function N(d) is also provided in any spreadsheet program. In Excel the function name is NORMSDIST, so using EXCEL (using interpolation for (43) and interpolation (.18), we find that

$N(.43) = .6664$
$N(.18) = .5714$

Finally, remember that with dividends $(\delta) = 0$ since $e^{0} = 1$
$S_0 e^{-\delta t} = S_0$,

Thus the value of the call option is

$C = 100\ (.6664) - 95e^{-10x0.25}\ (.5714)$
$C = 66.64 - 95\ (.9753)\ (.5714)$
$C = 66.64 - 52.94$

$C = \$13.70$

BLACK-SCHOLES OPTION VALUATION METHOD B/S - CALL OPTION

A	B	C	D	E	F	G	
139							
140	INPUT				OUTPUT		Excel Formula
141		Standard Deviation (σ) =	0.2783		d1 =	0.003	=(LN(D144/D145)+(D143+D146+D142*D141^2)*D142)/(D141*SQRT(D142))
142		Expiration (in years) (T) =	0.5		d2 =	-0.194	=+G141-D141*SQRT(D142)
143		Risk-Free Rate (Annual) (i) =	0.06		N(d1) =	0.501	=NORMSDIST(G141)
144		Stock Price (S) =	100		N(d2) =	0.423	=NORMSDIST(G142)
145		Exercise Price (X) =	105		B/S =	7.000	=+D144*EXP(-D146*D142)*G143-D145*EXP(-D143*D142)*G144
146		Dividend Yeild (annual) (δ) =	0				

Cummulative Normal Distribution Table

d	N(d)	d	N(d)	d	N(d)	d	N(d)	d	N(d)	d	N(d)
-3.00	0.0013	-1.48	0.0694	-0.56	0.2877	0.36	0.6406	1.28	0.8997	2.20	0.9861
-2.95	0.0016	-1.46	0.0721	-0.54	0.2946	0.38	0.6480	1.30	0.9032	2.22	0.9868
-2.90	0.0019	-1.44	0.0749	-0.52	0.3015	0.40	0.6554	1.32	0.9066	2.24	0.9875
-2.85	0.0022	-1.42	0.0778	-0.50	0.3085	0.42	0.6628	1.34	0.9099	2.26	0.9881
-2.80	0.0026	-1.40	0.0808	-0.48	0.3156	0.44	0.6700	1.36	0.9131	2.28	0.9887
-2.75	0.0030	-1.38	0.0838	-0.46	0.3228	0.46	0.6772	1.38	0.9162	2.30	0.9893
-2.70	0.0035	-1.36	0.0869	-0.44	0.3300	0.48	0.6844	1.40	0.9192	2.32	0.9898
-2.65	0.0040	-1.34	0.0901	-0.42	0.3372	0.50	0.6915	1.42	0.9222	2.34	0.9904
-2.60	0.0047	-1.32	0.0934	-0.40	0.3446	0.52	0.6985	1.44	0.9251	2.36	0.9909
-2.55	0.0054	-1.30	0.0968	-0.38	0.3520	0.54	0.7054	1.46	0.9279	2.38	0.9913
-2.50	0.0062	-1.28	0.1003	-0.36	0.3594	0.56	0.7123	1.48	0.9306	2.40	0.9918
-2.45	0.0071	-1.26	0.1038	-0.34	0.3669	0.58	0.7190	1.50	0.9332	2.42	0.9922
-2.40	0.0082	-1.24	0.1075	-0.32	0.3745	0.60	0.7257	1.52	0.9357	2.44	0.9927
-2.35	0.0094	-1.22	0.1112	-0.30	0.3821	0.62	0.7324	1.54	0.9382	2.46	0.9931
-2.30	0.0107	-1.20	0.1151	-0.28	0.3897	0.64	0.7389	1.56	0.9406	2.48	0.9934
-2.25	0.0122	-1.18	0.1190	-0.26	0.3974	0.66	0.7454	1.58	0.9429	2.50	0.9938
-2.20	0.0139	-1.16	0.1230	-0.24	0.4052	0.68	0.7517	1.60	0.9452	2.52	0.9941
-2.15	0.0158	-1.14	0.1271	-0.22	0.4129	0.70	0.7580	1.62	0.9474	2.54	0.9945
-2.10	0.0179	-1.12	0.1314	-0.20	0.4207	0.72	0.7642	1.64	0.9495	2.56	0.9948
-2.05	0.0202	-1.10	0.1357	-0.18	0.4286	0.74	0.7704	1.66	0.9515	2.58	0.9951
-2.00	0.0228	-1.08	0.1401	-0.16	0.4364	0.76	0.7764	1.68	0.9535	2.60	0.9953
-1.98	0.0239	-1.06	0.1446	-0.14	0.4443	0.78	0.7823	1.70	0.9554	2.62	0.9956
-1.96	0.0250	-1.04	0.1492	-0.12	0.4522	0.80	0.7881	1.72	0.9573	2.64	0.9959
-1.94	0.0262	-1.02	0.1539	-0.10	0.4602	0.82	0.7939	1.74	0.9591	2.66	0.9961
-1.92	0.0274	-1.00	0.1587	-0.08	0.4681	0.84	0.7995	1.76	0.9608	2.68	0.9963
-1.90	0.0287	-0.98	0.1635	-0.06	0.4761	0.86	0.8051	1.78	0.9625	2.70	0.9965
-1.88	0.0301	-0.96	0.1685	-0.04	0.4840	0.88	0.8106	1.80	0.9641	2.72	0.9967
-1.86	0.0314	-0.94	0.1736	-0.02	0.4920	0.90	0.8159	1.82	0.9656	2.74	0.9969
-1.84	0.0329	-0.92	0.1788	0.00	0.5000	0.92	0.8212	1.84	0.9671	2.76	0.9971
-1.82	0.0344	-0.90	0.1841	0.02	0.5080	0.94	0.8264	1.86	0.9686	2.78	0.9973
-1.80	0.0359	-0.88	0.1894	0.04	0.5160	0.96	0.8315	1.88	0.9699	2.80	0.9974
-1.78	0.0375	-0.86	0.1949	0.06	0.5239	0.98	0.8365	1.90	0.9713	2.82	0.9976
-1.76	0.0392	-0.84	0.2005	0.08	0.5319	1.00	0.8413	1.92	0.9726	2.84	0.9977
-1.74	0.0409	-0.82	0.2061	0.10	0.5398	1.02	0.8461	1.94	0.9738	2.86	0.9979
-1.72	0.0427	-0.80	0.2119	0.12	0.5478	1.04	0.8508	1.96	0.9750	2.88	0.9980
-1.70	0.0446	-0.78	0.2177	0.14	0.5557	1.06	0.8554	1.98	0.9761	2.90	0.9981
-1.68	0.0465	-0.76	0.2236	0.16	0.5636	1.08	0.8599	2.00	0.9772	2.92	0.9982
-1.66	0.0485	-0.74	0.2296	0.18	0.5714	1.10	0.8643	2.02	0.9783	2.94	0.9984
-1.64	0.0505	-0.72	0.2358	0.20	0.5793	1.12	0.8686	2.04	0.9793	2.96	0.9985
-1.62	0.0526	-0.70	0.2420	0.22	0.5871	1.14	0.8729	2.06	0.9803	2.98	0.9986
-1.60	0.0548	-0.68	0.2483	0.24	0.5948	1.16	0.8770	2.08	0.9812	3.00	0.9987
-1.58	0.0571	-0.66	0.2546	0.26	0.6026	1.18	0.8810	2.10	0.9821	3.02	0.9987
-1.56	0.0594	-0.64	0.2611	0.28	0.6103	1.20	0.8849	2.12	0.9830	3.04	0.9988
-1.54	0.0618	-0.62	0.2676	0.30	0.6179	1.22	0.8888	2.14	0.9838	3.06	0.9989
-1.52	0.0643	-0.60	0.2743	0.32	0.6255	1.24	0.8925	2.16	0.9846	3.08	0.9990
-1.50	0.0668	-0.58	0.2810	0.34	0.6331	1.26	0.8962	2.18	0.9854	3.10	0.9990